

# Two-particle fermionic quantum walks on a cycle graph

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In this paper we study quantum walks of electrons on a graph. This graph is composed of Si quantum dots arranged in a circle. Electrons can tunnel between neighbouring dots and interact via Coulomb interaction. We show that this mutual repulsion leads to entanglement. Fermionic entanglement dynamics is evaluated by several measures. Current detectors are considered in the paper as a noisy environment. We show that depending on the noise parameters fermionic entanglement can be annihilated, preserved, created for a certain time or re-created after annihilation.

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## I. INTRODUCTION

Quantum walks are quantum counterparts of classical random walks. But unlike classical walker, quantum walker's current state is described by a superposition of positions. Quantum walks is expected to have implications for various fields, for instance, as an element for quantum computing, quantum algorithms [1] or for understanding of the efficient energy transfer in biomolecules for photosynthesis [2]. In our paper we study quantum walks as an advanced tool for quantum information processing.

There are a lot of theoretical and experimental results in the field of quantum walks [3, 4]. But walks with multiple identical walkers, both the non-interacting and interacting cases, are relatively unexplored.

We investigate the system of coupled quantum dots that form a circle. Quantum dots are promising elements of a quantum computer. It was shown that quantum dots could be used to encode quantum information [5–8]. Quantum dot qudit consists of quantum dots with one electron connected by tunneling. Dots themselves are formed from the two-dimensional electron gas by field of gates and these dots are controlled by potentials on gates. As a result we have a qudit basis states  $|0\rangle$ ,  $|1\rangle$ , ... and  $|N\rangle$ , which describe the localization of an electron in 0-th, 1-st or  $N$ -th place, respectively. The position of an electron is measured by current detectors. Detectors are placed near quantum dots, if there is no charge in a dot then there is a current in a detector. An electron in a quantum dot creates a potential barrier, so the current in a nearest detector is zero [9].

In our paper we study a circle of quantum dots described above, but instead of one electron in this scheme we investigate two electrons. First, we analyse the dynamics of non-interacting particles, then we study interacting electrons.

## II. NON-INTERACTING INDISTINGUISHABLE PARTICLES

We consider a cycle graph with  $N$  nodes and two identical electrons in these nodes (Fig. 1). In this model electrons are indistinguishable because their wave functions overlap in space. The wave function of electrons is factorized in the form of  $\Psi(\mathbf{r}, t) = \psi(\mathbf{r}, t)\chi(t)$  and is antisymmetric under particle exchange. A pair of electrons can be combined to form a state with total spin one  $\chi = (\alpha|\downarrow\downarrow\rangle + \beta|\uparrow\uparrow\rangle + \gamma(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)/\sqrt{2})/\sqrt{3}$ , or a state with total zero spin  $(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)/\sqrt{2}$ . In this paper we study triplet states, i.e. symmetric spin part and antisymmetric coordinate part of wave function.

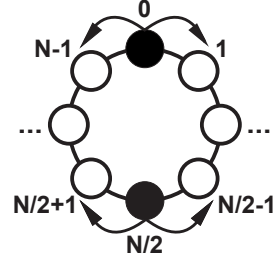


FIG. 1. The scheme of two-particle quantum walk on a cycle graph with  $N$  nodes. Electrons are initially in 0-th and  $N/2$ -th nodes.

We write the coordinate part of spin-1/2 fermionic wave function in the form of

$$\psi(\mathbf{r}) = \sum_{m,k=0}^{N-1} \omega_{mk} f_m^\dagger f_k^\dagger |0\rangle, \quad \omega^T = -\omega, \quad 2\text{Tr}(\omega\omega^\dagger) = I, \quad (1)$$

$f_m^\dagger$  is the fermionic creation operator and  $|0\rangle$  is the vacuum state [10] for an electron on site  $m$ ,  $N$  is the number of quantum dots in the cycle.

Electrons can tunnel between neighbouring places with tunnelling amplitude  $\Omega$  according to the hopping Hamil-

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tonian:

$$H = \hbar\Omega \sum_{i,j=0}^{N-1} (|x_{i+1}, y_j\rangle \langle x_i, y_j| + |x_i, y_j\rangle \langle x_{i+1}, y_j| + |x_j, y_{i+1}\rangle \langle x_j, y_i| + |x_j, y_i\rangle \langle x_j, y_{i+1}|), \quad (2)$$

where  $x_i$  and  $y_i$  are coordinates of first and second indistinguishable particles respectively.  $\Omega$  is responsible for the barrier height between single quantum dots. The spin part of the wave function  $\chi$  remains constant. We find the evolution of pure state from Schrödinger equation  $i\hbar\partial\psi/\partial t = H\psi$ , the unitary evolution of density matrix is  $\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$ . This unitary operator converts the antisymmetric fermionic wave function to antisymmetric one. The amplitudes of states  $|ii\rangle$  are set to zero.

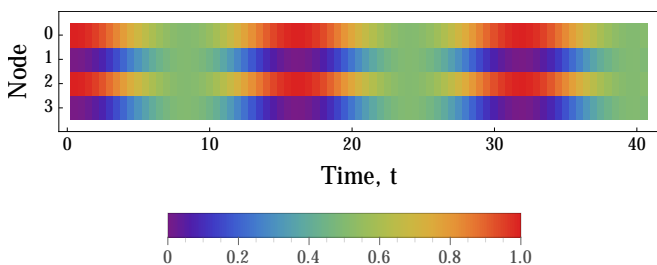


FIG. 2. Electron density in the  $i$ -th node vs. time. The initial state is  $(|02\rangle - |20\rangle)|\uparrow\uparrow\rangle/\sqrt{2}$ . The period of quantum walk is equal to  $\pi/\Omega$ . In these simulations we put  $\Omega = 0.1$  and  $N = 4$ .

Fig. 2 shows charge density dynamics in a system of 4 nodes. We consider spin state  $\chi(t) = |\uparrow\uparrow\rangle$  because one can do this in experiment by turning on the strong magnetic field. The initial coordinate state is  $(|02\rangle - |20\rangle)/\sqrt{2}$ . This state describes two indistinguishable electrons initially placed in the 0-th and 2-nd nodes.

The same probability distribution dynamics will have an electron which is initially in a superposition of  $|0\rangle$  and  $|2\rangle$  coordinate states. The probability of finding this single electron in certain node is half of the probability of finding one of the electrons. And in general, quantum walk of non-interacting particles are equivalent to one-particle walk, whose dynamics was studied in [9, 11, 12].

### III. INTERACTING INDISTINGUISHABLE ELECTRONS

Let us now turn to the case of two identical fermions which interact due to Coulomb interaction. Because of the repulsion electrons can not be in the same and neighbouring places. The hopping Hamiltonian with the restriction of being in the same and neighbouring sites of

the cycle graph is

$$H = \hbar\Omega \sum_{\substack{i,j=0 \\ j \neq i, i \pm 1, i+2}}^{N-1} (|x_{i+1}, y_j\rangle \langle x_i, y_j| + |x_i, y_j\rangle \langle x_{i+1}, y_j| + |x_j, y_{i+1}\rangle \langle x_j, y_i| + |x_j, y_i\rangle \langle x_j, y_{i+1}|). \quad (3)$$

Coulomb interaction between two electrons is a resource of quantum entanglement. Without this interaction there is no entanglement dynamics. The restriction of being in the same node with the same spin can not contribute to the entanglement because this is just the property of fermionic wave function.

The qualification and quantification of entanglement between several subsystems is one of the most important issues in quantum information theory. To describe entanglement of electrons we can not use the usual definition of entanglement of distinguishable particles, because for identical particles the Hilbert space no longer has a tensor product structure. The Hilbert space of two electrons is an antisymmetric product, not a direct product [13, 14].

#### A. Fermionic entanglement measures

To define entanglement of the indistinguishable fermions one can use Slater rank [10, 15, 16]. Slater rank is the minimum number of Slater determinants. This number is analogous to the Schmidt rank for the distinguishable case. Fermions are separable iff Slater rank is equal to one. That is quantum entanglement arise in a pure state if there is no single-particle basis such that a given state of electrons can be represented as a single Slater determinant

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_1(\mathbf{r}_1) & \psi_2(\mathbf{r}_1) \\ \psi_1(\mathbf{r}_2) & \psi_2(\mathbf{r}_2) \end{vmatrix}. \quad (4)$$

These correlations are the analogue of quantum entanglement in separated systems and are essential for quantum information processing in non-separated systems. The definition of fermionic entanglement using Slater rank is generalized for mixed states  $\rho = \sum_i p_i |\psi\rangle_i^{r_i} \langle\psi|_i^{r_i}$ , where  $r_i$  denotes the the Slater rank of the pure state  $|\psi\rangle_i^{r_i}$ . Mixed states can be characterized by their Slater number which is the minimal Slater rank required to generate them. By definition, the Slater number  $k = \min r_{\max}$ , where  $r_{\max}$  is the maximum Slater rank within a decomposition, and the minimum is taken over all decompositions.

An entanglement criterion for states of two fermions can be formulated in terms of the Von Neumann entropy [10, 17, 18]

$$S_{\text{vN}}(\rho_1) = -\text{Tr}(\rho_1 \ln \rho_1) = -\sum_j \lambda_j \ln \lambda_j, \quad (5)$$

$\lambda_j$  are eigenvalues of the single-particle reduced density matrix  $\rho_1$ . A pure state has Slater rank one iff  $S_{\text{vN}}(\rho_1) = \ln 2$ .

For two-particle mixed  $\rho$  [19]

$$E_{\text{vN}}(\rho) = S_{\text{vN}}(\rho_1) - S_{\text{vN}}(\rho) - \ln 2, \quad (6)$$

is positive the state  $\rho$  is necessarily entangled. If a fermionic state is pure, then  $E_{\text{vN}}(\rho) = S_{\text{vN}}(\rho_1) - \ln 2$ .

One can also use linear entropy  $S_{\text{L}}(\rho_1) = 1 - \text{Tr}\rho_1^2$  to quantify entanglement. A pure state has Slater rank one iff  $S_{\text{L}}(\rho_1) = 1/2$ . In general, a pure state of  $n$  identical fermions is separable if and only if the purity of the reduced density matrix  $\rho_1$  is equal to  $1/n$  [19–21]. That is the state has Slater rank one iff  $\text{Tr}\rho_1^2 = 1/n$ . And if  $1/d \leq \text{Tr}\rho_1^2 < 1/n$  then the state is entangled,  $d = 2N \geq n$  is the dimension of the single particle state space.

In the case of two-particle fermionic mixed density matrix  $\rho$  [19]

$$E_{\text{L}}(\rho) = S_{\text{L}}(\rho_1) - S_{\text{L}}(\rho) - \frac{1}{2} = \text{Tr}\rho^2 - \text{Tr}\rho_1^2 - \frac{1}{2}, \quad (7)$$

is positive the state  $\rho$  is necessarily entangled. For pure states we get the relation  $E_{\text{L}}(\rho) = S_{\text{L}}(\rho_1) - 1/2$ .

### B. Interacting electrons simulations

Let us consider the initial state, described by the wave function

$$\Psi(\mathbf{r}) = \psi(\mathbf{r}) \left( \alpha |\downarrow\downarrow\rangle + \beta |\uparrow\uparrow\rangle + \gamma (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) / \sqrt{2} \right) / \sqrt{3}. \quad (8)$$

For each  $N$  the state is entangled or separable according to the  $\psi(\mathbf{r})$ ,  $\alpha$ ,  $\beta$  and  $\gamma$ . For example, states  $\Psi = (|02\rangle - |20\rangle) |\uparrow\uparrow\rangle / \sqrt{2}$  and  $\Psi = (|02\rangle - |20\rangle) (|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) / 2\sqrt{2}$  are separable, because we get  $S_{\text{L}}(\rho_1) = 1/2$ ,  $E_{\text{L}}(\rho) = E_{\text{vN}}(\rho) = 0$  and  $S_{\text{vN}}(\rho_1) = \ln 2$ . The state  $\Psi = (|02\rangle - |20\rangle) (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) / 2$  is entangled, because we get  $E_{\text{L}}(\rho) = 1/4$ ,  $S_{\text{L}}(\rho_1) = E_{\text{vN}}(\rho) = \ln 2$  and  $S_{\text{vN}}(\rho_1) = 2 \ln 2$ .

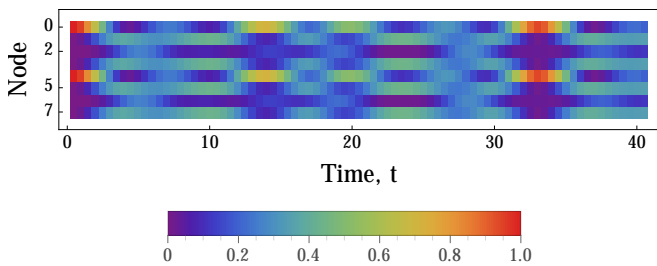


FIG. 3. Electron density in the  $i$ -th node vs. time. In these simulations we put  $\Omega = 0.3$  and  $N = 8$ . The initial state is  $(|04\rangle - |40\rangle) |\uparrow\uparrow\rangle / \sqrt{2}$ . Mutual repulsion between electrons is taken into account. At time  $t \approx 33$  the charge density in the 0 node is less than one, it is about 0.94.

Fig. 3 shows charge density dynamics in a system of 8 quantum dots arranged in a closed ring. We can see

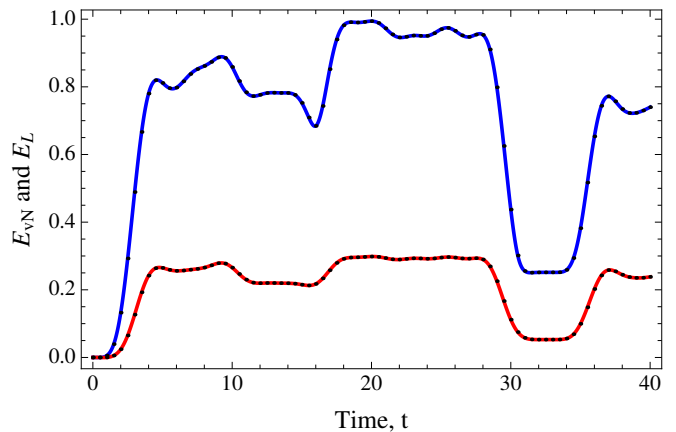


FIG. 4. Von Neumann (blue) and linear (red) entropy. In these simulations we put  $\Omega = 0.3$  and  $N = 8$ . The initial fermionic state is  $(|04\rangle - |40\rangle) |\uparrow\uparrow\rangle / \sqrt{2}$ . At time  $t \approx 30$  there is a sudden drop of entanglement with the local minimum of entanglement at time  $t \approx 33$ . At small  $t$  Von Neumann entropy contains more information about fermionic entanglement than linear entropy.

that there is no period of walks, only partial repetition. We calculate  $E_{\text{vN}}(\rho)$  and  $E_{\text{L}}(\rho)$  using Eqs. (6) and (7) respectively. An evolution of entanglement according to entanglement sufficient conditions is shown in Fig. 4. It can be seen that two electrons are initially separable, but after a time they become entangled. This entanglement is suddenly drops then there is a partial repetition of the initial separable state at time  $t \approx 33$  (Fig. 3). Von Neumann entropy (6) sufficient condition detects entanglement earlier, than linear entropy (7). It is worth noting that Von Neumann entropy contains more information about fermionic entanglement than linear entropy according to the sufficient condition.

### IV. DECOHERENCE

Let us turn to the description of electron quantum walks in the presence of environment. One can measure the two qudit state using current detectors. It was shown [9] that these detectors are the sources of white (depolarizing) noise. We use the density matrix formalism to describe the system of electrons in environment. The evolution of the density matrix because of the white noise is  $\dot{\rho} = -\Gamma(\rho - \rho_M \otimes \rho_{\text{spin}})$ ,  $\Gamma \geq 0$  is a relaxation rate. Solving this equation we get the following:

$$\rho(t) = e^{-\Gamma t} \rho(0) + (1 - e^{-\Gamma t}) \rho_M \otimes \rho_{\text{spin}}, \quad (9)$$

with maximally mixed fermionic state matrix  $\rho_M \otimes \rho_{\text{spin}}$  of  $N^2$  dimension, where  $N$  is the number of nodes. The spin part of the density matrix does not change during the evolution, that is the spin part in Eq. (9) is  $\rho_{\text{spin}} = \text{Tr}_{\text{coord}} \rho = \text{Tr}_{\text{coord}} \rho(0)$ . The maximally mixed state of

the coordinate part is

$$\rho_M = \frac{1}{N^2 - N} \sum_{j=1}^{N-1} \sum_{i=0}^{j-1} (|i, j\rangle - |j, i\rangle) (\langle i, j| - \langle j, i|). \quad (10)$$

The maximal mixed state of the coordinate part for 2 and 3 nodes are the following:

$$\rho_M^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho_M^{(3)} = \frac{1}{6} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

The general expression for fermionic mixed state evolution is

$$\rho(t) = e^{-iHt/\hbar} (e^{-\Gamma t} \rho(0) + (1 - e^{-\Gamma t}) \rho_M \otimes \rho_{\text{spin}}) e^{iHt/\hbar}. \quad (12)$$

The density matrix  $\rho(t)$  is antisymmetric, because it describes two indistinguishable fermions. The same matrix can be obtained by using an ‘‘ordinary’’ density matrix  $\rho'$  that describes distinguishable particles. To convert ordinary density matrix to antisymmetric one  $\rho = A\rho'A^\dagger$ ,  $A = \frac{N^2 - N}{\sqrt{2}} \rho_M$ .

The decoherence described by Eq. (9) can lead to errors in quantum information stored in two qudits. We use the measure of decoherence  $D$  [22, 23] to quantify the amount of errors. By definition,  $D = \sup_{\rho_{\text{in}}} \|\rho_{\text{out}} - \rho_{\text{in}}\|$ , where the operator norm of the matrix  $X$  is given by  $\|X\| = \max_{x \in \text{spec}(X)} |x|$ , by  $\text{spec}(X)$  denote the spectrum of operator  $X$  [24].  $D$  can be thought of as a probability of a false result obtained by current detectors. Although the decoherence process is well studied for global depolarizing noise (9), the difference between distinguishable particles and indistinguishable fermions is not clear. The measure of decoherence for distinguishable particles is

$$D_{\text{dist}}(N) = (1 - e^{-\Gamma t}) \left(1 - \frac{1}{N^2}\right). \quad (13)$$

We get the measure of decoherence for fermions

$$D_{\text{fermi}}(N) = (1 - e^{-\Gamma t}) \left(1 - \frac{2}{N^2 - N}\right). \quad (14)$$

Comparing Eqs. (13) and (14) we can see that depolarizing noise introduce less errors in fermionic system with arbitrary number of nodes.

The main goal of this paper is to study fermionic entanglement dynamics. To get the time evolution of entanglement we calculate  $E_{\text{vN}}(\rho)$  from Eq. (6). Simulations with different noise parameters are shown in Fig. 5.

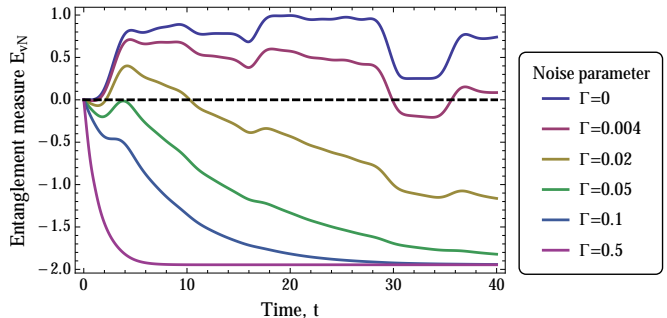


FIG. 5. Fermionic entanglement evolution. We put  $\Omega = 0.3$ ,  $N = 8$  and the initial state is  $(|04\rangle - |40\rangle)|\uparrow\uparrow\rangle/\sqrt{2}$ .

If the noise is strong enough, the entanglement between two electrons will probably not appear, as it is shown in Fig. 5 for  $\Gamma = 0.1$ . This situation differs from  $\Gamma = 0$  case where entanglement is created because electrons repel approaching each other. An intermediate case is depicted in Fig. 5 ( $\Gamma = 0.004$ ). The decoherence parameter  $\Gamma = 0.004$  is not strong enough and entanglement is created after a certain time, but then the process of decoherence starts to prevail over the process of fermionic quantum walks. The entanglement is annihilated and re-created and, finally, the noise will annihilate entanglement.

## V. CONCLUSION

In the paper we considered fermionic quantum walks. Quantum walks is a more natural process comparing to the gate creation. It was shown before that one can do arbitrary quantum operations using only particles free propagation [3]. One way to realize quantum walks algorithms is to use silicon quantum dots that form a cycle graph. We showed that using this structure one can entangle electrons. The value of fermionic entanglement was calculated using measures (6) and (7) that were introduced recently [19]. We presented the decoherence of fermionic state model in Section IV. The decoherence is because of the depolarizing noise from detectors [9]. Analytical expressions 13 and 14 for measure of decoherence were obtained and compared in the case of distinguishable and identical particles. But this noise can change not only the coherence of the state, but also fermionic entanglement. Fig. 5 shows computed entanglement dynamics for different noise levels. It can be seen that entanglement behaves differently. Strong noise ( $\Gamma > 0.05$ )

annihilate entanglement and weak noise ( $\Gamma < 0.0001$ ) preserve entanglement after its creation due to electrons repulsion. And there are intermediate cases: entanglement is created for a certain time ( $\Gamma = 0.02$ ) or annihilated for a certain time ( $\Gamma = 0.004$ ). One also could make appropriate conditions in which entanglement will be created and annihilated periodically during the quantum walks. This dynamics will be obtained without any

additional manipulations with electrons and could find several applications in quantum information science.

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