Continuous-time quantum walk of two interacting fermions on a cycle graph

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ABSTRACT

We study a continuous-time quantum walk of interacting fermions on a cycle graph. By finding analytical solutions and simulating the dynamics of two fermions we observe a diverse structure of entangled states of indistinguishable fermions. The relation between entanglement of distinguishable qutrits and indistinguishable electrons is observed. Restrictions imposed by the symmetry of a cycle graph are derived. Possible realization of a quantum walk in an array of semiconductor quantum dots is discussed.

Keywords: Quantum walks, cycle graph, interacting electrons, fermionic entanglement, quantum dots

1. INTRODUCTION

A quantum walk[1,2] of two interacting identical particles is studied in this paper. This walk, which deterministically generates entanglement between initially spatially separated separable particles, is performed on a cycle graph with N vertices. As a realistic physical implementation of a quantum walk[3] on a cycle we consider dynamics of electrons in an array of semiconductor quantum dots.[4–7] In Ref. 8 it was shown that one can use electrons, which are placed in semiconductor quantum dots, for encoding quantum information. Errors in such an encoding are mostly caused by an interaction of electrons with acoustic phonons,[9] but they can be corrected by means of quantum error correction algorithms.[10,11]

A continuous-time quantum walk on a cycle graph that we consider is based on the model studied in Ref. 12. Here, however, we consider two electrons that due to their identity introduce additional features in the quantum state dynamics. A general state of two electrons will be described as a superposition of its basis states $|\,\psi^{(i,j)}\,\rangle$:

$$|\,\psi(t)\,\rangle = \sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1} \alpha_{ij}(t)\,|\,\psi^{(i,j)}\,\rangle\,, \tag{1}$$

where the basis state

$$|\,\psi^{(i,j)}\,\rangle = \frac{1}{\sqrt{2}}\left(|\,i\,\rangle\,|\,j\,\rangle - |\,j\,\rangle\,|\,i\,\rangle\right) \tag{2}$$

is an antisymmetric wave function that describes two fermions in positions i and j. Here for the sake of simplicity we assume that spin part of two fermion wave function is symmetric under permutation. In particular, this is the case when high constant magnetic field is applied. Then all spins are aligned and spin flips are suppressed. The coefficients α_{ij} are the amplitudes of a fermionic quantum state with $\sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1}|\alpha_{ij}|^2 = 1$ normalization condition. By using notations in Eq. 1 we assume that only amplitudes $\alpha_{ij}(t)$ with $j > i$ contribute to the superposition, which is justified because the considered wave function is antisymmetric under permutation of positions of fermions.

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A probability to find an electron in a particular place k can be written as a function of amplitudes of a fermionic wave function

$$\lambda_k(t) = \sum_{j=k+1}^{N-1} |\alpha_{kj}(t)|^2 + \sum_{i=0}^{k-1} |\alpha_{ik}(t)|^2 . \tag{3}$$

Due to the indistinguishability of two fermions, both fermions have the same probability to be detected in a particular place k. Note that the probability $\lambda_k(t)$ can also be treated as a population of particles in a place k, because $\lambda_k(t)$ quantifies a number of electrons that can be found on average in a place k by doing measurements at time t. For that reason $\lambda_k(t)$ is also a charge distribution in the case of electrons in semiconductor quantum dots.

The Hamiltonian that governs the evolution of $|\,\psi(t)\,\rangle$ is the following

$$H = \hbar\Omega \sum_{i=0}^{N-1} \sum_{j=i+2}^{N+i-3} |\,(i+1) \bmod N\,\rangle\,\langle\, i \bmod N\,| \otimes |\,j\,\rangle\,\langle\,j\,| + |\,i\,\rangle\,\langle\,i\,| \otimes |\,(j+1) \bmod N\,\rangle\,\langle\,j \bmod N\,| + \text{H.c.}, \tag{4}$$

where Ω is a tunneling frequency. This Hamiltonian is time-independent and approximates propagation of two electrons affected by a Coulomb repulsion between them.[13] As it was shown in Ref. 13, entangled states of electrons can be created by free evolution under the Hamiltonian in Eq. 4. These entangled states can be used for quantum information purposes and represent two-qudit (two quantum d-level systems) states. Here we continue our investigation by studying a variety of entangled states, which are possible to create via a quantum walk preparation procedure.

2. RESULTS

A continuous-time quantum walk under consideration is performed by electrons that are initially prepared in the

$$|\,\psi(0)\,\rangle = \frac{1}{\sqrt{2}} \left(|\,0\,\rangle\,|\,K\,\rangle - |\,K\,\rangle\,|\,0\,\rangle\right) \tag{5}$$

state with $K = N/2$ for a circle with an even number of nodes N. The specific choice of the initial state is based on the ability to experimentally prepare the state by separately placing two electrons in most distinct from each other quantum dots. This initialization procedure assures that initially electrons do not influence each others state. And although we describe the initial state with a fermionic wave function from Eq. 5, the state itself represents a classical separable state of two particles. However, starting from $|\,\psi(0)\,\rangle$ and evolving under the Hamiltonian H, electrons immediately become entangled.[13]

In this paper we consider a special state of electrons during their quantum walk, in which electrons are uniformly distributed over the circle. That means that at a certain time τ each electron can be found anywhere on the cycle graph with an equal probability, i.e.

$$\lambda_j(\tau) = \frac{1}{N}. \tag{6}$$

As it will be shown below, the state of two uniformly distributed fermions has a number of interesting properties. In general, however, it is not guarantied that time τ that satisfies Eq. 6 exists for any size of a circle and for a specified initial condition. Nonetheless, as it is shown below, for several small sizes of the graph we were able to find time τ approximately by numerically simulating the dynamics of interacting electrons. Furthermore, for the case of $N = 6$ by using an analytical solution[13] we explicitly demonstrate the existence of the uniform charge distribution. For time

$$\tau = \frac{1}{\sqrt{6}\Omega} \left(\arccos\left(\sqrt{3} - 2\right) + 2\pi n\right),\ n \in Z \tag{7}$$

we obtain

$$\begin{aligned} |\,\psi\,\rangle_6 = &\left(\frac{2}{\sqrt{3}} - 1\right)|\,\psi^{(0,3)}\,\rangle + \left(\frac{1}{\sqrt{3}} - 1\right)\left(|\,\psi^{(1,4)}\,\rangle - |\,\psi^{(2,5)}\,\rangle\right) - \\ &- i\sqrt{\frac{2}{\sqrt{3}} - 1}\left(|\,\psi^{(0,2)}\,\rangle + |\,\psi^{(0,4)}\,\rangle + |\,\psi^{(1,3)}\,\rangle - |\,\psi^{(3,5)}\,\rangle\right). \end{aligned} \tag{8}$$

This state of two fermions at this time is depicted in Fig 1, obtained by numerically solving the Schrödinger equation. Fig. 1(a) shows a probability distribution $\lambda_j(\tau)$ for all $0 \le j \le 5$ at time τ specified by Eq. 7. One can see that all λ_j are approximately equal, that is particles are distributed uniformly over the cycle graph with 6 vertices. This uniform distribution is, however, does not represent a state with fully random and independent positions of electrons. On the contrary, the state with uniformly distributed electrons is highly entangled, which can be seen by computing a fermionic entanglement measure for the state in Eq. 8. Here for entanglement quantification we use the fermionic concurrence:[14]

$$C_{\mathrm{F}} = \sqrt{\frac{2N}{N-2}\left(\frac{1}{2} - \mathrm{Tr}\rho_1^2\right)}, \tag{9}$$

where ρ_1 is the reduced one-particle density matrix. For the state in Eq. 8 we obtain $C_{\mathrm{F}} \approx 0.79$. Other entanglement measures for the case of two fermions, which are suitable for our problem, are discussed in Ref. 15.

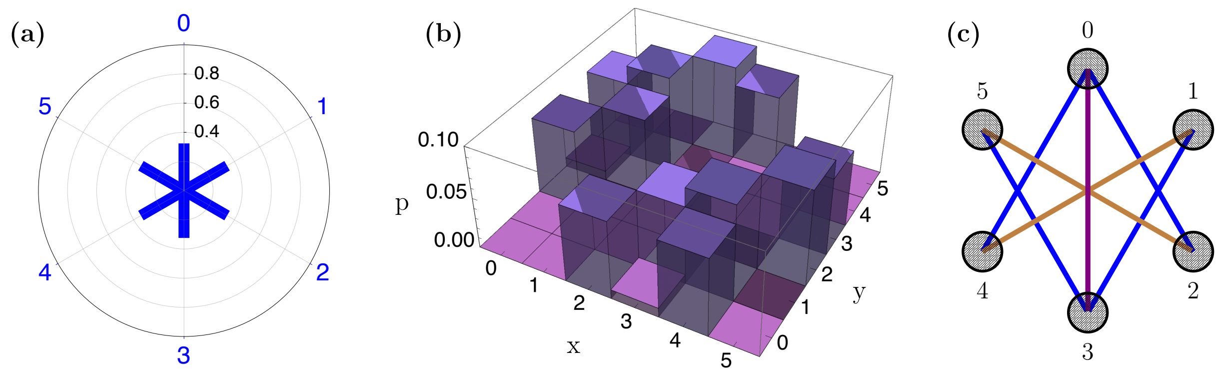

Figure 1. A quantum state of two electrons at time $t\Omega = 3.317$, which approximately corresponds to the case of $t\Omega = \frac{1}{\sqrt{6}} \arccos\left(\sqrt{3} - 2\right) + \frac{2\pi}{\sqrt{6}}$. (a) Charge population λ_k in each vertex $0 \le k \le 5$. Electrons are uniformly distributed over the circle. (b) Correlation matrix, which corresponds to probability of finding a pair of electrons in positions x and y. (c) Types of entanglement. Each line connects two possible positions of electrons. Color coding corresponds to different terms in Eq. 8.

Fig. 1(b), where probabilities $p(x, y)$ to find electrons in positions x and y are plotted, depicts all possible combinations of positions in which electrons can reside. Because electrons are indistinguishable, the coincidence matrix is symmetric, i.e. $p(y, x) = p(x, y)$. In Fig. 1(c) we schematically summarize all the possibilities to see two electrons: there are 7 different combinations of positions, each shown with a line that connects these positions. Some combinations, due to the symmetry, must have the same probability to be observed, and are shown with the same color.

As it was shown in Ref. 13, one can define two qudits, i.e. two d-level of quantum information, in the described system. In the case shown in Fig. 1, one can define two qutrits (3-level systems) by assigning positions 5, 0 and 1 to the first fermion, and positions 2, 3 and 4 to the second fermion. In other words, if one observer has an access to the upper part of the circle, and the second observer has an access to the lower part of the circle, then they share an entangled pair of fermions. The high degree of entanglement, which we quantify by C_F, can be verified by local measurements implemented by individual observers and the subsequent communication between them. Without communication the observers will not be able to distinguish an entangled state from a mixed state of a uniformly distributed charge.

Note that after defining two qutrits, particles become distinguishable (upper and lower electrons), and a measure of entanglement for two 3-level systems should be used. By analogy with C_F, one can consider concurrence

$$C = \sqrt{\frac{3}{2}\left(1 - \mathrm{Tr}\left(\widetilde{\rho_1}^2\right)\right)}, \tag{10}$$

where

$$\widetilde{\rho_1} = \frac{\sum_{i,j=5,0,1} |i\rangle \langle i| \rho_1 |j\rangle \langle j|}{\mathrm{Tr}\left(\sum_{i,j=5,0,1} |i\rangle \langle i| \rho_1 |j\rangle \langle j|\right)} \tag{11}$$

is a reduced density matrix of the upper qutrit. It is important that for the case of two qutrits we obtain $C = C_F$. Hence we conclude that entanglement of indistinguishable electrons is fully transferred to entanglement of two qudits.

The state with equally distributed electrons for circle with $N = 8$ nodes is obtained numerically for time $\tau \approx 8.1/\Omega$ and is shown in Fig. 2. Similar to Fig. 1(a), in Fig. 2(a) one can see the charge population, which is approximately equal in all vertices. Fig. 2(b) depicts the probabilities to detect electrons in positions x and y at time $\tau \approx 8.1/\Omega$. Fig. 2(c) schematically demonstrates all non-zero terms in the superposition of Eq. 1 by connecting positions of two electrons with a line. Lines of the same color correspond to equal amplitudes in the superposition and represent one class of entangled states. One can see that there more different classes of entanglement (two additional colors: red and green) in comparison to the case of $N = 6$. This is due to additional possibilities enabled by dots 2 and 6 in Fig. 2(c). The value of entanglement of the state in Fig. 2 is estimated by using the fermionic concurrence in Eq. 9, which gives a value of $C_F \approx 0.88$. Note that one can reduce the case of $N = 8$ to the smaller size $N = 6$ by doing a postselection: the final state will be conditioned on zero electrons being detected in dots 2 and 6.

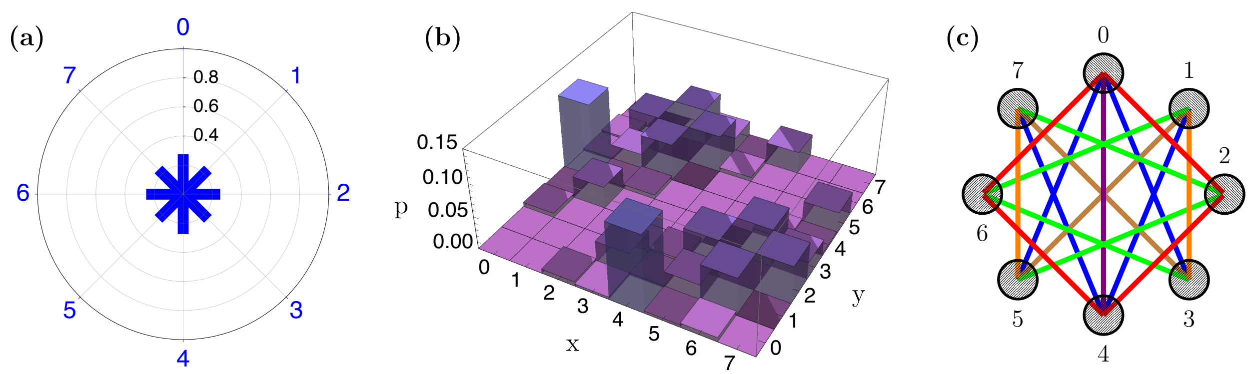

Figure 2. A quantum state of two electrons at time $t\Omega = 8.1$. (a) Charge population λ_k in each vertex $0 \le k \le 7$. Electrons are uniformly distributed over the circle. (b) Correlation matrix, which corresponds to probability of finding a pair of electrons in positions x and y. (c) Types of entanglement. Each line connects two possible positions of electrons. Different colors correspond to unequal amplitudes of fermionic wave function.

Next we show that it is impossible to detect electrons arranged in a horizontal line, i.e. nodes i and $(N - i)$ for $0 < i < N$ cannot be occupied at the same time. Due to the symmetry of the initial state, for each non-zero $\alpha_{i,j}$ amplitude, there exists a non-zero $\alpha_{N-j,N-i} = -\alpha_{i,j}$. For $j = N - i$ we get $\alpha_{i,N-i} = -\alpha_{i,N-i}$, hence

$$\alpha_{i,N-i} = 0 \tag{12}$$

and $|\alpha_{i,N-i}|^2 = 0$. Apart from this restriction and Coulomb repulsion restriction, there are no other rules that prohibit two fermions to be detected in arbitrary positions on a graph.

One can see that obviously these restrictions apply to the specific sizes of a cycle graph with $N = 6$, $N = 8$ and $N = 10$ vertices, shown in Fig. 1, Fig. 2, and Fig. 3. Moreover, the state with equally distributed electrons has all possible combinations of electron positions, which one can see from Fig. 1(c) and Fig. 2(c). Fig. 3(c) depicts in dotted gray all the restrictions imposed by Coulomb repulsion and Eq. 12. The value of entanglement of the state in Fig. 3 is estimated by using the fermionic concurrence, which gives a value of $C_F \approx 0.88$. The value is similar to the one from the case of $N = 8$.

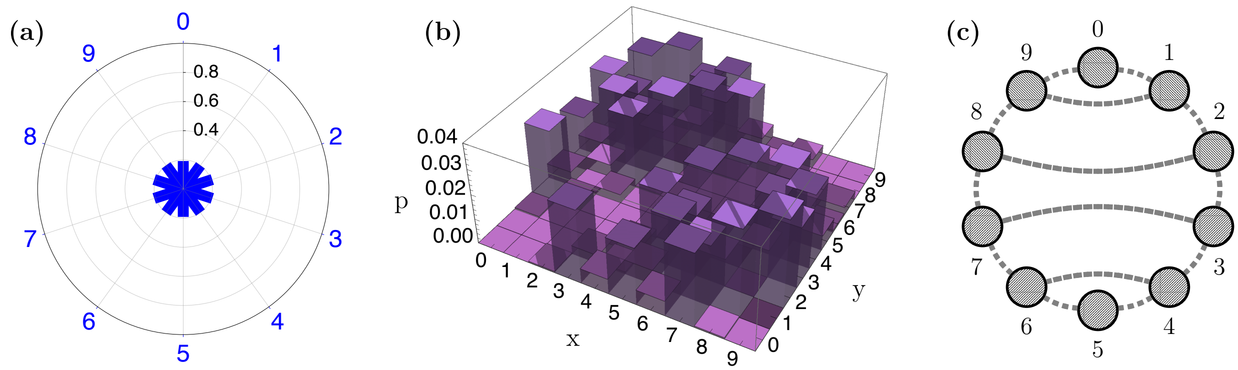

Figure 3. A quantum state of two electrons at time $t\Omega = 6.9$. (a) Charge population λ_k in each vertex $0 \leq k \leq 9$. Electrons are uniformly distributed over the circle. (b) Correlation matrix, which corresponds to probability of finding a pair of electrons in positions x and y. (c) Dashed gray lines connect pairs of dots, which are impossible to occupy at the same time.

As we discussed before, an entangled state of two fermions can be shared between two observers. In realistic scenario, the shared state will be subjected to decoherence, which can be modelled by a depolarizing noise. This noise, nevertheless, will not change the charge distribution over the circle, it will stay uniform. However the noise will create a mixture of positions instead of a coherent superposition, which will be noticeable to individual observers. The observers can detect errors if they detect two electrons in a horizontal line, i.e. nodes i and $(N-i)$ for $0 < i < N$.

3. CONCLUSION

Two fermion quantum walks were investigated. Dynamics of fermionic entanglement arising due to interaction was obtained. By considering the case of uniformly distributed electrons, we identified classes of entangled states, which depend on the size of the cycle graph. Additional restrictions on possible classes of entangled states imposed by the symmetry of the graph and the initial conditions were derived. For a specific size of the cycle graph, we showed that fermionic entanglement can be transferred to entanglement of two distinguishable qutrits without any loss of quantum correlations. The results presented open new perspectives for multi-fermion quantum walks experiments and creation of truly entangled multi-fermion states.

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